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Singularity confinement test for ultradiscrete equations with parity variables

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Abstract
We present a new ultradiscretization method which does not require that the solutions of the discrete equation have a fixed sign. We construct an ultradiscrete analogue of the singularity confinement test using this method and thereby propose an integrability test for ultradiscrete equations.

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1. Introduction

Diverse models of natural phenomena are expressed in terms of differential equations, difference equations and cellular automata, most of which are nonlinear and cannot be solved. Only rarely we can find equations that are integrable, in which case there exist special properties such as symmetries, conserved quantities or exact solutions. An intense research has been carried out over the decades to find out what characterizes integrability and a simple way to detect it.

One integrability criterion for ordinary differential equations is the existence of Painlevé property. An equation whose movable singularities are at most poles is said to satisfy the Painlevé property, in which case the equation is predicted to be integrable. This property was considered by Fuchs for first-order nonlinear ordinary differential equations. Later, Painlevé and Gambier extended the idea to second-order equations and found the celebrated Painlevé equations. A subsequent application of this criterion is the Painlevé conjecture, which is an integrability detector for partial differential equations.

In the realm of the discrete system, the notion of integrability is more ambiguous. The singularity confinement test (SC) has been proposed in [1, 2] as an integrability detector for discrete equations. It is predicted that a singularity of an integrable discrete equation cancels out after a finite number of iterations without loss of information on the initial condition, and this property is the discrete analogue of Painlevé property. Although reports of counterexamples in [3–5] indicate that passing SC does not guarantee integrability, SC is useful for the discovery of integrable discrete equations and is considered to be closely related to integrability.

Recent developments in ultradiscretization [6] have led to the research of integrability detectors for cellular automata. Ultradiscretization is a procedure for transforming a discrete equation into an equation in an ultradiscrete system, in which dependent variables also take discrete values. In general, to apply this procedure, we first replace each variable (or parameter) x of a discrete equation by a new variable X defined by $x = e^{\frac{X}{\varepsilon}}$ where $\varepsilon > 0$ is a parameter. Then in the limit $\varepsilon \to +0$, addition and multiplication of the original variables are replaced by max functions and addition of the new variables, respectively, by employing the identity

$$\lim_{\varepsilon \to +0} \varepsilon \log \left(e^{\frac{X}{\varepsilon}} + e^{\frac{Y}{\varepsilon}} \right) = \max(X, Y) \tag{1.1}$$

and exponential laws. One advantage of working with an ultradiscrete equation is that the properties of the discrete equation are emphasized after taking an ultradiscrete limit. However, ultradiscretization requires that the discrete variables have a fixed sign, and this restriction does not allow one to transfer all properties of the discrete equation into the ultradiscrete system. An ultradiscrete analogue of SC proposed in [7] and studied in [8] is no exception.

Ultradiscretizations without the fixed-sign constraint have been discussed in [9–11]. In [9, 10], a new variable X defined by $x = \sinh(X/\varepsilon)$ is introduced for ultradiscretizing discrete equations. In this paper, we extend this idea and present a new method that can be used for constructing an ultradiscrete analogue of SC. Based on our findings, we propose a criterion for integrability of ultradiscrete equations.

2. Ultradiscretization with parity variables

To illustrate our new ultradiscretization method, consider the first-order discrete equation for x_n:

$$x_{n+1} = a x_n, \tag{2.1}$$

where a is a nonzero parameter. The general solution of (2.1) is given by

$$x_n = a^n x_0. \tag{2.2}$$

Depending on the sign and magnitude of a, (2.2) is a solution growing exponentially or decaying to zero, either monotonically or with oscillation.

For standard ultradiscretization of (2.1), we impose the constraint $a, x_n > 0$. We then introduce a new parameter A and a new dependent variable X_n defined by $a = e^{\frac{A}{\varepsilon}}, x_n = e^{\frac{X_n}{\varepsilon}}$ respectively. Finally, taking the ultradiscrete limit $\varepsilon \to +0$ of the resulting expression transforms (2.1) into

$$X_{n+1} = A + X_n. \tag{2.3}$$

The solution to (2.3) is given by

$$X_n = nA + X_0, \tag{2.4}$$

which is precisely the expression obtained by ultradiscretizing (2.2). The ultradiscrete solution (2.4) preserves the monotonic behaviour of (2.2), but the oscillatory behaviour is missing due to the positive-value constraint.

To transfer all qualitative behaviours of (2.2) into the ultradiscrete system, we adopt the following ultradiscretization procedure instead. We first introduce parity variables $\sigma, \sigma_n \in \{-1, 1\}$ and amplitude variables $b, y_n > 0$, and write $a = \sigma b, x_n = \sigma_n y_n$. We further introduce a function $s : \{-1, 1\} \to \{0, 1\}$ defined by

$$s(\tau) := \begin{cases} 1 & (\tau = 1) \\ 0 & (\tau = -1), \end{cases} \tag{2.5}$$

and write $\sigma = s(\sigma) - s(-\sigma)$, $\sigma_n = s(\sigma_n) - s(-\sigma_n)$. Collecting non-negative terms to each side of equality and substituting $b = e^{\frac{B}{\varepsilon}}$, $y_n = e^{\frac{Y_n}{\varepsilon}}$, we can take the ultradiscrete limit of the resulting expression to obtain the following implicit equation for σ_n and Y_n:

$$\begin{aligned}\max[&Y_{n+1} + S(-\sigma_{n+1}), B + Y_n + \max\{S(\sigma) + S(\sigma_n), S(-\sigma) + S(-\sigma_n)\}] \\ &= \max[Y_{n+1} + S(\sigma_{n+1}), B + Y_n + \max\{S(\sigma) + S(-\sigma_n), S(-\sigma) + S(\sigma_n)\}], \end{aligned} \qquad (2.6)$$

where the function $S : \{-1, 1\} \to \{0, -\infty\}$ is defined by

$$S(\tau) := \begin{cases} 0 & (\tau = 1) \\ -\infty & (\tau = -1). \end{cases} \qquad (2.7)$$

We refer to this new ultradiscretization method outlined here and the corresponding ultradiscrete equation (2.6) as ultradiscretization with parity variables and ultradiscrete equation with parity variables, respectively. Since taking $\sigma = \sigma_{n+1} = \sigma_n = 1$ reduces (2.6) to (2.3), an ultradiscrete equation with parity variables is a generalization of the standard ultradiscrete equation.

To find the solution to (2.6), it is more convenient to rewrite (2.6) as the following pair of explicit equations:

$$\sigma_{n+1} = \sigma\sigma_n, \qquad (2.8)$$

$$Y_{n+1} = B + Y_n. \qquad (2.9)$$

Denoting each point of (2.6) by $X_n := (\sigma_n, Y_n)$, we obtain the following solution:

$$X_n = (\sigma^n\sigma_0, nB + Y_0). \qquad (2.10)$$

The case $\sigma = -1$ produces an oscillatory solution, meaning that (2.10) describes all qualitative properties that (2.2) contains.

3. Ultradiscrete singularity confinement test

We start our discussions on ultradiscretization of SC with the notion of singularity in the ultradiscrete system. Recall that, given a second-order discrete equation $x_{n+1} = f(x_n, x_{n-1})$, a point x_n is a singular point if $\partial x_{n+1}/\partial x_{n-1} = 0$ for a generic initial condition x_{n-1}, and a point x_{n+k} $(k = 1, 2, 3, \ldots)$ is a singularity if $\partial x_{n+k}/\partial x_{n-1} = 0$. In a similar way, we define the singularity of an ultradiscrete equation as follows.

Definition 1. *A point $X_n = (\sigma_n, Y_n)$ is a singular point of a second-order ultradiscrete equation with parity variables if $X_{n+1} = (\sigma_{n+1}, Y_{n+1})$ is indeterminate for a generic initial condition $X_{n-1} = (\sigma_{n-1}, Y_{n-1})$. A point $X_{n+k} = (\sigma_{n+k}, Y_{n+k})(k = 1, 2, 3, \ldots)$ is a singularity if it is dependent on the indeterminacy X_{n+1}.*

A singularity of a discrete equation generally propagates indefinitely unless, in the case of an integrable equation, the initial condition is recovered by the appearance of an indeterminate form in the subsequent iterations. As we shall see in the following, an ultradiscrete analogue of this phenomenon is the appearance of a new indeterminacy at the expense of the original singularity.

Definition 2. *A second-order ultradiscrete equation with parity variables is said to pass ultradiscrete singularity confinement test (uSC) if, iterating with an initial condition X_{n-1} and a singular point X_n, the points X_{n+k} and X_{n+k+1} are independent of the singularity X_{n+1} for some $k \geqslant 2$.*

As an illustrative example, we consider the following discrete equation [8]:

$$x_{n+1}x_n^{\gamma}x_{n-1} = ax_n + 1. \tag{3.1}$$

If $\gamma = 0, 1, 2$, (3.1) is integrable in the sense that it belongs to the Quispel–Roberts–Thompson (QRT) system and passes SC. In what follows, we consider only the case $a > 0$.

Ultradiscretizing (3.1) with parity variables for $\gamma = 2$ leads to the following equation:

$$\begin{aligned}\max[Y_{n+1} + Y_{n-1} + \max\{S(\sigma_{n+1}) + S(\sigma_{n-1}), S(-\sigma_{n+1}) + S(-\sigma_{n-1})\},\quad A - Y_n + S(-\sigma_n)]\\ = \max[Y_{n+1} + Y_{n-1} + \max\{S(\sigma_{n+1}) + S(-\sigma_{n-1}), S(-\sigma_{n+1}) + S(\sigma_{n-1})\},\\ A - Y_n + S(\sigma_n), -2Y_n].\end{aligned} \tag{3.2}$$

Provided $X_n \neq (-1, -A)$, (3.2) admits the following pair of explicit equations:

$$\sigma_{n+1} = \sigma_{n-1}\left[\frac{\sigma_n}{2}\{1 + \mathrm{sgn}(A + Y_n)\} + \frac{1}{2}\{1 - \mathrm{sgn}(A + Y_n)\}\right], \tag{3.3}$$

$$Y_{n+1} = \max(A - Y_n, -2Y_n) - Y_{n-1}, \tag{3.4}$$

where the signum function $\mathrm{sgn} : \mathbb{R} \to \{-1, 0, 1\}$ is defined by

$$\mathrm{sgn}(Z) := \begin{cases} 1 & (Z > 0)\\ 0 & (Z = 0)\\ -1 & (Z < 0). \end{cases} \tag{3.5}$$

On the other hand, (3.3) and (3.4) are invalid if $X_n = (-1, -A)$, in which case we find by substitution into (3.2) that $X_{n+1} = (\sigma_{n+1}, Y_{n+1})$ where $\sigma_{n+1} = \pm 1$ and $Y_{n+1} \leqslant 2A - Y_{n-1}$. To see how the singularity propagates, we iterate with the initial condition $X_0 = (\sigma_0, F)$ and the singular point $X_1 = (-1, -A)$. It turns out that there exist various singularity patterns depending on the value of F. In what follows, we present only the results obtained for $A > 0$ (the case $A < 0$ is similar).

The time evolution up to X_6 for $F > 3A$ is given below:

$$\begin{aligned}&X_0 = (\sigma_0, F)\quad X_1 = (-1, -A)\quad X_2 = (\sigma_2, Y_2),\quad \sigma_2 = \pm 1,\quad Y_2 \leqslant 2A - F\\ &X_3 = (-1, A - 2Y_2)\quad X_4 = (-\sigma_2, Y_2)\quad X_5 = (-1, -A)\\ &X_6 = (\sigma_6, Y_6),\quad \sigma_6 = \pm 1,\quad Y_6 \leqslant 2A - Y_2.\end{aligned} \tag{3.6}$$

In this case, the condition on F determines the time evolution beyond X_2 uniquely. Singularity is confined from X_2 to X_4, a new indeterminacy appears at X_6 and the original singularity disappears in the subsequent iterations. The time evolution (3.6) therefore passes uSC. In what follows, we iterate up to the appearance of a new indeterminacy.

The time evolution for $-A < F \leqslant 3A$ is as follows:

$$\begin{aligned}&X_0 = (\sigma_0, F)\quad X_1 = (-1, -A)\\ &X_2 = (\sigma_2, Y_2),\quad \sigma_2 = \pm 1,\quad Y_2 < -A\\ &X_3 = (-1, A - 2Y_2)\quad X_4 = (-\sigma_2, Y_2)\quad X_5 = (-1, -A)\\ &X_6 = (\sigma_6, Y_6),\quad \sigma_6 = \pm 1,\quad Y_6 \leqslant 2A - Y_2\end{aligned} \tag{3.7}$$

$$\begin{aligned}&X_0 = (\sigma_0, F)\quad X_1 = (-1, -A)\\ &X_2 = (\sigma_2, Y_2),\quad \sigma_2 = \pm 1,\quad -A < Y_2 \leqslant 2A - F\\ &X_3 = (-\sigma_2, 2A - Y_2)\quad X_4 = (-1, -A)\\ &X_5 = (\sigma_5, Y_5),\quad \sigma_5 = \pm 1,\quad Y_5 \leqslant Y_2\end{aligned} \tag{3.8}$$

In this case, there exist two singularity patterns depending on the choice of Y_2. The choice $Y_2 < -A$ gives the same pattern as that of (3.6). If $Y_2 > -A$, a new indeterminacy appears at X_5 at the expense of the original singularity, meaning that this time evolution also passes uSC.

Finally, the time evolution for $F \leqslant -A$ is as follows:

$$\begin{aligned}
&X_0 = (\sigma_0, F) \quad X_1 = (-1, -A) \quad X_2 = (\sigma_2, Y_2),\\
&\sigma_2 = \pm 1, \quad Y_2 < -A \quad X_3 = (-1, A - 2Y_2)\\
&X_4 = (-\sigma_2, Y_2) \quad X_5 = (-1, -A)\\
&X_6 = (\sigma_6, Y_6), \quad \sigma_6 = \pm 1, \quad Y_6 \leqslant 2A - Y_2
\end{aligned} \tag{3.9}$$

$$\begin{aligned}
&X_0 = (\sigma_0, F) \quad X_1 = (-1, -A) \quad X_2 = (\sigma_2, Y_2),\\
&\sigma_2 = \pm 1, \quad -A < Y_2 < 3A \quad X_3 = (-\sigma_2, 2A - Y_2)\\
&X_4 = (-1, -A) \quad X_5 = (\sigma_5, Y_5),\\
&\sigma_5 = \pm 1, \quad Y_5 \leqslant Y_2
\end{aligned} \tag{3.10}$$

$$\begin{aligned}
&X_0 = (\sigma_0, F) \quad X_1 = (-1, -A) \quad X_2 = (\sigma_2, Y_2),\\
&\sigma_2 = \pm 1, \quad 3A < Y_2 \leqslant 2A - F \quad X_3 = (-\sigma_2, 2A - Y_2)\\
&X_4 = (\sigma_2, -4A + Y_2) \quad X_5 = (-1, 3A)\\
&X_6 = (-\sigma_2, 2A - Y_2) \qquad \vdots
\end{aligned} \tag{3.11}$$

In this case, we obtain a new pattern (3.11) for $Y_2 > 3A$ in which the singularity propagates indefinitely. This singularity pattern emerges for the following reason. Recall that the singularity pattern for (3.1) with $\gamma = 2$ is obtained by iterating with $x_0 = f$ and $x_1 = -1/a + \epsilon$ where $|\epsilon| \ll 1$. The perturbation expansion breaks down if we allow $|\epsilon|$ to be large, and (3.11) corresponds to such a case.

Similar analysis has been carried out for (3.1) with $\gamma = 0, 1$ and the following integrable discrete equation [12]:

$$x_{n+1} = \frac{ax_n + 1}{(a + x_n)x_n x_{n-1}}, \tag{3.12}$$

and it has been confirmed that, for any initial condition, we obtain similar results as that of above.

On the other hand, (3.1) with $\gamma = 3$ is not a QRT mapping and does not pass SC. In this case, ultradiscretization with parity variables gives the following equation:

$$\begin{aligned}
&\max[Y_{n+1} + Y_{n-1} + \max\{S(\sigma_{n+1}) + S(\sigma_{n-1}), S(-\sigma_{n+1}) + S(-\sigma_{n-1})\}, -3Y_n + S(-\sigma_n)]\\
&\quad = \max[Y_{n+1} + Y_{n-1} + \max\{S(\sigma_{n+1}) + S(-\sigma_{n-1}), S(-\sigma_{n+1}) + S(\sigma_{n-1})\},\\
&\quad\quad A - 2Y_n, -3Y_n + S(\sigma_n)],
\end{aligned} \tag{3.13}$$

whose associated pairs of explicit equations for $X_n \neq (-1, -A)$ are given by

$$\sigma_{n+1} = \sigma_{n-1}\left[\frac{\sigma_n}{2}\{1 - \mathrm{sgn}(A + Y_n)\} + \frac{1}{2}\{1 + \mathrm{sgn}(A + Y_n)\}\right], \tag{3.14}$$

$$Y_{n+1} = \max(A - 2Y_n, -3Y_n) - Y_{n-1}. \tag{3.15}$$

If $X_n = (-1, -A)$, we are led to the singularity $X_{n+1} = (\sigma_{n+1}, Y_{n+1})$ where $\sigma_{n+1} = \pm 1$ and $Y_{n+1} \leqslant 3A - Y_{n-1}$. The time evolution for $F > 4A > 0$ is given below:

$$
\begin{aligned}
&X_0 = (\sigma_0, F) \quad X_1 = (-1, -A)\\
&X_2 = (\sigma_2, Y_2), \quad \sigma_2 = \pm 1, \quad Y_2 \leqslant 3A - F\\
&X_3 = (-\sigma_2, A - 3Y_2) \quad X_4 = (\sigma_2, -A + 5Y_2)\\
&X_5 = (-1, 2A - 12Y_2) \quad X_6 = (\sigma_2, -2A + 19Y_2)\\
&X_7 = (-\sigma_2, 4A - 45Y_2) \quad \vdots
\end{aligned}
\tag{3.16}
$$

The singularity propagates indefinitely, which shows that (3.16) does not pass uSC. It has also been confirmed that time evolutions for other values of F exhibit similar behaviours.

Based on our observations, we claim that integrable ultradiscrete equations have the following property in common:

Claim 3. *A second-order ultradiscrete equation with parity variables is integrable if, iterating with an initial condition $X_{n-1} = (\sigma_{n-1}, Y_{n-1})$ and a singular point $X_n = (\sigma_n, Y_n)$, there exists a constant C such that, for any X_{n-1}, every time evolution obtained by choosing $Y_{n+1} < C$ (or possibly $Y_{n+1} > C$) passes uSC.*

4. Concluding remarks

In this paper, we have presented a new ultradiscretization procedure, which we call ultradiscretization with parity variables, by the introduction of a parity variable σ_n and an amplitude variable Y_n. The resulting ultradiscrete equation is in implicit form but its solution is compatible with that of the discrete equation. We insist that an ultradiscrete equation with parity variables is a generalization of the standard ultradiscrete equation and our new method can be applied to a wider class of discrete equations.

The crucial point of SC is how the information on the initial condition is recovered. In the ultradiscrete system, this corresponds to the appearance of a new indeterminacy at the expense of the original singularity. Although we have studied only a handful of equations in this paper, we insist that such a phenomenon is common to all integrable ultradiscrete equations. Just as in the discrete case, further applications of uSC may include non-autonomizing an autonomous ultradiscrete equation or testing an ultradiscrete soliton equation for singularity confinement.

It must be pointed out, however, that uSC is nothing but an ultradiscrete analogue of SC. Consequently, whether our new integrability criterion is necessary or sufficient for integrability remains questionable. It may be important to study the concept of growth for ultradiscrete equations [8, 13]. We conclude this paper by insisting that, just as in the discrete case, uSC is closely related to integrability of ultradiscrete equations.

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